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PRAGMATIC AND DIALOGIC INTERPRETATIONS OF BI-INTUITIONISM
Part I

Abstract. We consider a "polarized" version of bi-intuitionistic logic [5, 2, 6, 4] as a logic of *assertions and hypotheses* and show that it supports a "rich proof theory" and an interesting categorical interpretation, unlike the standard approach of C. Rauszer's *Heyting-Brouwer logic* [28, 29], whose categorical models are all partial orders by Crolard's theorem [8]. We show that P. A. Melliès notion of *chirality* [21, 22] appears as the right mathematical representation of the mirror symmetry between the intuitionistic and co-intuitionistc sides of polarized bi-intuitionism. Philosophically, we extend Dalla Pozza and Garola's *pragmatic interpretation of intuitionism* as a logic of assertions [10] to bi-intuitionism as a logic of *assertions and hypotheses*. We focus on the logical role of illocutionary forces and justification conditions in order to provide "intended interpretations" of logical systems that classify inferential uses in natural language and remain acceptable from an intuitionistic point of view. Although Dalla Pozza and Garola originally provide a constructive interpretation of intuitionism in a classical setting, we claim that some conceptual refinements suffice to make their "pragmatic interpretation" a *bona fide* representation of intuitionism. We sketch a *meaning-asuse* interpretation of co-intuitionism that seems to fulfil the requirements of Dummett and Prawitz's justificationist approach. We extend the Brouwer-Heyting-Kolmogorov interpretation to bi-intuitionism by regarding co-intuitionistic formulas as types of the evidence for them: if *conclusive evidence* is needed to justify assertions, only a *scintilla of evidence* suffices to justify hypotheses.

Keywords: bi-intuitionism; categorical proof theory; justificationism; meaning-asuse; speech-acts theory.

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1. Introduction. A mathematical prelude

The mathematical case study of this paper is a variant of Cecylia Rauszer's *bi-intuitionistic* logic [28, 29] (called *Heyting–Brouwer* logic by Rauszer) and the relations between the two parts that can be identified within it, namely *intuitionistic* logic, on one hand, and *co-intuitionistic* (also known as *anti-intuitionistic* or *dual-intuitionistic*), on the other hand.[1] Our main goal is to identify, among the mathematical models of bi-intuitionism, those which may be regarded as its *intended interpretations*. The quest for an *intended interpretation* of a formal system often arises when several mathematical structures have been proposed to characterise an informal, perhaps vague notion and furthermore more unfamiliar and vaguer extensions arise by analogy or by opposition: here philosophical analysis may be invoked to assess which formal systems belong to *logic*, in the sense that they do capture actual forms of human reasoning, rather than to pure or applied mathematics.

It is very appropriate to ask such a question about bi-intuitionism: following Rauszer's approach researchers in this area usually define bi-intuitionism by extending intuitionistic logic with the connective of subtraction $C \smallsetminus D$, to be read as "C *excludes* D", which in algebraic terms is the left adjoint to disjunction in the same way as implication is the right adjoint to conjunction (see the rules in (1.1) below). This pair of adjunctions establishes a *duality* between the core *minimal fragments* of intuitionism and co-intuitionism, namely, intuitionistic conjunction and implication with a logical constant for validity, on one hand, and co-intuitionistic disjunction and subtraction with invalidity, on the other.

However when bi-intuitionistic logic is defined in this way essential properties of the model theory and proof theory of co-intuitionism and of bi-intuitionism no longer hold. Recently Tristan Crolard ([8, 9]) developed bi-intuitionistic proof theory by adding rules for subtraction to *classical* proof theory and then introduced restrictions to characterize the constructive fragment. A reason for this choice is that if bi-intuitionism is regarded as an extension of intuitionistic logic with the connective of subtraction, then the "intuitionistic status" of bi-intuitionism becomes unclear: it was probably E. G. K. López-Escobar [19] the first to notice

[1] We are mostly indebted with Paul-André Melliès for pointing at his work on dialogue chirality and at its relevance to our approach to bi-intuitionism. We are grateful for this insight that does clarify the nature of polarized bi-intuitionistic logic and the issue of its categorical models.

that *first order bi-intuitionistic* logic is not a conservative extension of first order intuitionistic logic,[2] since in the standard theory of first order bi-intuitionistic logic one can derive the following *intuitionistically invalid inference* (where x does not occur free in B):

$$\forall x(Ax \vee B) \vdash (\forall x.Ax) \vee B.$$

Now it it is well-known that any first order theory containing this formula is complete for the semantics of *constant domains*. Thus first order bi-intuitionistic logic is an intermediate system between classical and intuitionistic logic (see T. Crolard [8] for a clear and detailed account of this matter). It turns out that when topological and categorical models are taken into account, very serious problems emerge that make Rauszer's bi-intuitionism unsuitable as a framework for developing intuitionistic and co-intuitionistic model-theory and proof-theory.

1.1. Bi-Heyting algebras and Kripke models

The early model theory of bi-intuitionism, namely, *bi-Heyting algebras* and Kripke-style semantics is due to Cecylia Rauszer [28, 29].

DEFINITION 1.1. A Heyting algebra is a bounded lattice $\mathcal{A} = (A, \vee, \wedge, 0, 1)$ (namely, with *join* and *meet* operations, the least and greatest element), and with a binary operation, *Heyting implication* ($\rightarrow$), which is defined as the right adjoint to meet. A *co-Heyting algebra* is a lattice $\mathcal{C}$ such that its opposite $\mathcal{C}^{op}$ (reversing the order) is a Heyting algebra. $\mathcal{C}$ has structure $(C, \vee, \wedge, 1, 0)$ with an operation of *subtraction* ($\smallsetminus$) defined as the left adjoint of join. Thus we have the rules

$$\begin{array}{ccc} \textit{Heyting algebra} & \qquad & \text{co-Heyting algebra} \\ \dfrac{c \wedge b \ \leq \ a}{c \ \leq \ b \rightarrow a} & \qquad & \dfrac{a \ \leq \ b \vee c}{a \smallsetminus b \ \leq \ c} \end{array} \tag{1.1}$$

A *bi-Heyting* algebra is a lattice that has both the structure of Heyting and of a co-Heyting algebra.

DEFINITION 1.2 (Rauszer's Kripke semantics). Kripke models for bi-intuitionistic logic have the form $\mathcal{M} = (W, \leq, \Vdash)$ where the accessibility relation $\leq$ is reflexive and transitive, and the forcing relation "$\Vdash$" satisfies the usual conditions for $\vee$, $\wedge$, 0 and 1 and moreover

[2] We are grateful to Rodolfo Ertola Biraben for giving us this reference.

$$
\begin{array}{lll}
w \Vdash A \to B & \text{iff} & \forall w' \geq w.w' \Vdash A \text{ implies } w' \Vdash B; \\
w \Vdash A \smallsetminus B & \text{iff} & \exists w' \leq w.w' \Vdash A \text{ and } w' \nVdash B.
\end{array}
$$

Such conditions guarantee the *monotonicity* property for all bi-intuitionistic formulas. Informally, they could be explained by saying that implication has to hold in all possible worlds *"in the future of our knowledge"* and subtraction in some world *"in the past of our knowledge"*. In fact Rauszer's Kripke semantics for bi-intuitionistic logic is associated with a *modal translation* into (what is called today) *tensed* **S4**.

1.2. No categorical bi-intuitionistic theory of proofs

In the corpus of mathematical intuitionism very basic constructions are the Brouwer-Heyting-Kolmogorov interpretation, where formulas are interpreted as types of their proofs, and the Extended Curry-Howard correspondence between the typed λ-calculus, intuitionistic Natural Deduction and Cartesian Closed Categories, in the interpretation of William Lawvere [17]. Here model theory and proof theory meet at a new level, where also *categorical proof theory* plays an essential role. Indeed categorical proof theory is concerned not only with algorithm to establish the *provability* of formulas in given proof systems, but has also mathematical tools to characterize the *identity of proofs*. To quote the simplest example, the philosophical conjecture by Martin-Löf and Prawitz that Natural Deduction derivations reducing to the same normal form represent the same intuitive proof can be treated axiomatically and refined in terms of the functorial properties and natural equivalences in Cartesian Closed Categories. Such a mathematical study where the notion of a proof can be appropriately characterised in relation to significant aspects of computation may be called a *rich proof theory*.

How are these ideas extended from intuitionism to co-intuitionism and bi-intuitionism? Recent work in co-intuitionistic and bi-intuitionistic proof theory (starting from the notes in appendix to Prawitz [25]) exploits the formal symmetry between intuitionistic conjunction and implication, on one hand, and co-intuitionistic disjunction and subtraction, on the other, in various formalisms, the sequent calculus, as in Czermak [11] and Urbas [32], the display calculus by Goré [16] or natural deduction by Uustalu [33], see also [24]. Luca Tranchini [31] shows how to turn Prawitz Natural Deduction trees upside down, as it was done also by the first author in [5, 2, 6], who has also developed a computational interpretation and a categorical semantics for co-intuitionistic linear logic [6, 4].

But the most striking fact is a theorem by Tristan Crolard [8]:

Theorem 1.1. *If a Cartesian Closed Category has also the dual structure of a co-Cartesian Closed category, then it is a partial order.*

Thus for Rauszer's bi-intuitionistic logic we can no-longer have a *categorical theory of proofs*: between two objects there is at most one morphism. The outcome is devastating: there cannot be a "rich proof theory" for Rauszer bi-intuitionism by a simple notion of duality.

In this paper we explore a solution to this problem that has been suggested in [5, 2], namely, *"polarizing"* bi-intuitionistic logic so as to "keep the dual intuitionistic and co-intuitionistic parts separate", but connected by "mixed operators", most notably, negations.

1.3. Co-intuitionistic disjunction is "multiplicative"

A second results by Tristan Crolard [8] shows that intuitionistic dualities are not modelled in the naif way in the category **Set**. The category **Set** is an important model of intuitionism, as the adjunction between *categorical products*, given by cartesian products, and *exponents*, given by sets of functions, models the adjunction between *conjunction* and *implication*. By duality, a categorical model of *co-intuitionism* is based on the adjunction between *categorical coproducts* modelling *disjunction* and *co-exponents* modelling *subtraction*. But here there is a main difference between intuitionism and co-intuitionism: Tristan Crolard [8] shows that in the category **Set** the co-exponent of two non-empty sets does not exist. A proof of Crolard's lemma is given in Appendix A.1.

The reason for this failure lies in the fact that in **Set** co-products are given by *disjoint unions*; in logical terms, this corresponds to the fact that a proof of A *or* B is always either a proof of A or a proof of B; intuitionistic disjunction involves a choice between the disjuncts. Following Girard's classification of connectives in linear logic [13], it is the *additive form of intuitionistic disjunction* that makes it an unsuitable candidate as a right adjoint of subtraction.

The solution advocated in [4] is to take *multiplicative disjunction*, namely, J-Y. Girard's *par*, as basic for the co-intuitionistic consequence relation and construct a categorical model of *linear co-intuitionistic logic* in *monoidal categories*, where *co-exponents* modelling *subtraction*, are indeed the left adjoint of *co-products* modelling *par*. Thus we have categorical models of *multiplicative linear* intuitionistic and co-intuitionistic logic and we are left with the problem of extending such models to full bi-intuitionistic logic, rather than its linear part.

1.4. Bi-intuitionistic logic as a chirality

In our reformulation of bi-intuitionism as a *polarized* system the idea emerges of a logic where the intuitionistic and the co-intuitionistic sides remain separated and form what P-A. Melliès [21, 22] calls a *chirality*, i.e., a mirror symmetry between independently defined structures $(\mathcal{A}, \mathcal{B})$, rather than a pair $(\mathcal{C}, \mathcal{C}^{op})$ where one element is defined as the opposite of the other. More precisely, a *chirality* is an adjunction $L \dashv R$ between monoidal functors $L \colon \mathcal{A} \to \mathcal{B}$ and $R \colon \mathcal{B} \to \mathcal{A}$, where $\mathcal{A} = (\mathcal{A}, \wedge, \mathbf{true})$ and $\mathcal{B} = (\mathcal{B}, \vee, \mathbf{false})$, together with a monoidal functor $(\_)^* \colon \mathcal{A} \to \mathcal{B}^{op}$ that allows to give a "De Morgan representation of implication" in $\mathcal{A}$ through disjunction of $\mathcal{B}$. The notion of chirality applies both to linear bi-intuitionism and to full bi-intuitionism and it appears as the right mathematical framework to develop these logics. We sketch the proof-theoretic treatment corresponding to the categorical notion of chirality (see also Appendix A.2), but we shall not do the categorical construction here.

But linear logic and the consideration of the relations between *classical* and *intuitionistic* linear logic give us also the tools of *Chu's construction* [3], a method to produce models of classical multiplicative linear logic from a pair of models of intuitionistic multiplicative linear logic, namely, from a pair $(\mathcal{C}, \mathcal{C}^{op})$ of monoidal closed categories. A simple application of Chu's construction yields also models of bi-intuitionistic linear logic from a pair of monoidal closed categories. Conceptually, this is important because Chu's construction suggests a *dialogue semantics* of bi-intuitionism inspired by an abstract form of the *game semantics* for linear logic. It is also clear that the two sides of the interpretation are exactly mirror images, i.e., form a chirality in an obvious sense.

In the rest of this paper we discuss the conceptual aspects of our *pragmatic interpretation* of bi-intuitionism. Next we give a precise definition of the language of *polarized bi-intuitionistic* and of *linear polarized bi-intuitionistic* logic and of our *dialogue interpretation*. Finally in the appendix we recall the basic definitions of our "proof theoretic" *Chu's construction* and show how to produce the dialogue interpretations of linear intuitionism and linear co-intuitionism as mirror images, i.e., as a chirality.

2. Philosophical interpretations of co-intuitionism

An important contribution to a philosophical understanding of co-intuitionism has been given by Yaroslav Shramko [30]. Co-intuitionistic sentences are interpreted as *statements that have not yet been refuted*, thus evoking the status of scientific laws in Popper's epistemology. In this view universal empirical statements can never be *conclusively justified*, but can be refuted by cumulative evidence against them (if not by a single crucial experiment). A clear merit of this approach is to have pointed at *formal epistemology* as a large domain where co-intuitionistic logic can be usefully applied.
Granted that the hypothetical status of empirical laws opens the way for application of co-intuitionism to formal epistemology, a question arises about the interpretation of the co-intuitionistic consequence relation and of inferences in co-intuitionism. We may consider a relation of the following form:

$$H \vdash H_1, \ldots, H_n \tag{2.1}$$

to be read as

> **H.0:** *the disjunction of* $H_1, \ldots, H_n$ *may* justifiably *be taken as a hypothesis given that it is* justified *to take* H *as a hypothesis.*

Here we follow ideas of D. Prawitz [27] on the explanation of deductive inference and justification of inference rules and assume that a consequence relation should be explained not only in terms of validity in a Kripke-style semantics, namely, by saying that the disjunction of $H_1, \ldots, H_n$ is true in all possible world in which H is true, but also in terms of the *justification conditions* for the act of making hypotheses, namely, by explaining how the evidence giving sufficient grounds for making the hypothesis H would also give sufficient grounds for taking the disjunction of $H_1, \ldots, H_n$ as a hypothesis.

Thus assuming that we know what "sufficient grounds for making a hypothesis H" are and borrowing the notion of "effective method" from the Brouwer-Heyting-Kolmogorov interpretations of intuitionism, we may give an effective interpretation of (2.1) as follows:

> **H.1:** *there is a method* F *transforming sufficient evidence for regarding* H *as a* justified hypothesis *into sufficient evidence for regarding the disjunction of* $H_1, \ldots, H_n$ *as a* justified hypothesis.

Let's assume that the meaning of a co-intuitionistic statement H is *"H is a still un-refuted hypothesis"*: this seems to imply that a justification for taking H as a hypothesis is the fact that H has not been refuted. Also this presupposes that we do know what "sufficient grounds for refuting a hypothesis H" are. But denying a hypothesis is *asserting* its falsity and refuting a hypothesis is giving *conclusive grounds* for such a denial, in particular in the case of a mathematical statement a *proof* of the falsity of H. We are back in the well-known environment of the Brouwer-Heyting-Komogorov interpretation; an *effective* interpretation of the relation (2.1) is as follows:

> **H.2:** *there is a method F^{op} to transform evidence* refuting *all the hypotheses $H_1, \ldots, H_n$ into evidence* refuting *the hypothesis H.*

So what is the primary notion, that of *sufficient grounds for making a hypothesis H* (*evidence for H*) or that of *sufficient grounds for refuting H* (*evidence against H*)? Or do we need both notions?

We may expect a fundamental objection to taking **H.1** as primitive. Many would say that no matter how "evidence for a hypothesis" is defined, it is the business of empiric sciences and of probability theory, not of logic, to deal with it. Hypothetical reasoning is inferring assertable propositions from the assumption that some propositions are assertable; strictly speaking, logic can only be about the *refutation* of hypotheses, as in the medieval practice of disputation [1].

Mathematical reasoning is mainly assertive and its proofs provide the paradigmatic notion of "conclusive evidence". However, other areas of deductive reasoning, including legal argumentation [15, 7], are about statements for which only non-conclusive degrees of evidence are available. We cannot discuss such applications here. Let us explore co-intuitionism as a logic of hypotheses and take the elementary expressions of our object language to represent types of hypotheses and the interpretation **H.1** of the consequence relation as primitive, as in work by the first author, [5, 2, 6, 4] aiming at a "rich proof theory" for co-intuitionism and bi-intuitionism. One should recognize that such mathematical treatment has focussed on the duality between intuitionism and co-intuitionism in order to design Gentzen systems, term assignments and categorical proof-theory for co-intuitionism. One should not underestimate the difficulty of taking co-intuitionism "on its own" and **H.1** as primitive: there is only one degree of *conclusive evidence*, but there are uncountably many degrees of partial evidence, according to probability theory. Do we need infinitely valued logics here?[3]

Remark 2.1. From a mathematical point of view it would seem appropriate, given a hypothesis $\mathcal{H}\,p$ and the evidence we have to justify it, to assign a probability to $\mathcal{H}\,p$ expressing our degree of confidence in its validity. This could be done in a classical probabilistic model, or in a Bayesian setting. In the literature on linear logic we find work by P. Lincoln, J. Mitchell and A. Scedrov [18] with a stochastic interaction semantics modelling proof search in *multiplicative and additive linear logic* **MALL**; in that framework logical connectives are interpreted as probabilistic operators. But to construct a model of co-intuitionistic logic we would need a translation into linear logic *with exponential operators* ? and ! and we do not have a stochastic interpretation of them. How should we interpret the consequence relation in **H.0**, **H.1** and **H.2** in terms of probability functions? Are probabilities assigned according to proof-search algorithms appropriate in our case? We cannot speculate about such questions here.

It is clear to us that a proper treatment of hypotheses both in applied contexts such as legal or medical evidence or formal epistemology and in a purely theoretical context does eventually require a probabilistic framework.[4] However it is also clear that if we regard *making the hypothesis that* p ($\mathcal{H}\,p$) as an illocutionary act in natural language, then the act of *asserting that* p *is true with probability* $Pr(p)$" conveys more information and is justified by much stronger conditions that simply making a hypothesis; nevertheless a *probabilistic modelling* of $\mathcal{H}\,p$ would certainly be adequate in any application to common sense reasoning.

2.1. A meaning-as-use justification of co-intuitionism?

If co-intuitionism is to stand as a logic on its own, representing informal practices of common sense reasoning, the question may be asked whether its inferential principles are compatible with the basic tenets of intuitionistic philosophy: mathematical duality may not suffice to justify such compatibility. A way to answer such a question and dispel doubts

[3] Of course this problem is already there in intuitionism, if we take into account what counts as *evidence against* an assertion, not only the *evidence for* it.

[4] Carlo Dalla Pozza in private conversation has often pointed out that hypotheses in science are best modelled in a Bayesian framework rather than through purely logical methods.

about its constructive nature is to give a *meaning-as-use* interpretation of co-intuitionism in the sense of Michael Dummett [12] and Dag Prawitz (in a sequence of papers from [26] to [27]).

Here we recall the main ingredients of such an interpretation. We take Natural Deduction rules of introduction and elimination for subtraction (in sequent-style form) and check that they satisfy the *inversion principle* (see [25]).

$$\searrow\text{-}intro\ \frac{H \vdash \Gamma, C \qquad D \vdash \Delta}{H \vdash \Gamma, C \searrow D, \Delta}$$

$$\searrow\text{-}elim\ \frac{H \vdash \Delta, C \searrow D \qquad C \vdash D, \Upsilon}{H \vdash \Delta, \Upsilon}$$

Notice that in the $\searrow$-*elimination* rule the evidence that D may be derivable from C given by the *right premise* has become *inconsistent with the hypothesis* $C \searrow D$ in the left premise; in the conclusion we drop D and we *set aside* the evidence for the inconsistent alternative. We may think that such evidence is not destroyed, but rather stored somewhere for future use.

If the *left premise* of $\searrow$-*elimination*, deriving the disjunction of $C \searrow D$ with Δ from H, has been obtained by a $\searrow$-*introduction*, then such an occurrence of $C \searrow D$ is a *maximal formula* and the pair of *introduction/elimination* rules can be eliminated: using the *removed evidence* for D derivable from C (*right premise* of the $\searrow$-*elim.*) we can conclude that the disjunction of $\Delta_1, \Delta_2, \Upsilon$ is derivable from H. This is, in a nutshell, the principle of normalization (or *cut-elimination*) for subtraction.

$$\searrow\text{-E}\ \frac{\searrow\text{-I}\ \dfrac{\overset{d_1}{H \vdash \Gamma, C} \qquad \overset{d_3}{D \vdash \Delta}}{H \vdash \Gamma, \Delta, C \searrow D} \qquad \overset{d_2}{C \vdash D, \Upsilon}}{H \vdash \Gamma, \Delta, \Upsilon}$$

reduces to

$$subst\ \frac{subst\ \dfrac{\overset{d_1}{H \vdash \Gamma, C} \qquad \overset{d_2}{C \vdash D, \Upsilon}}{H \vdash \Gamma, D, \Upsilon} \qquad \overset{d_3}{D \vdash \Delta}}{H \vdash \Gamma, \Delta, \Upsilon}$$

Now suppose d_1 and d_3 are simply *assumptions* (in the sequent form of axioms). Then we have the following reduction:

$$
\smallsetminus\text{-E}\;\dfrac{\smallsetminus\text{-I}\;\dfrac{C \vdash C \qquad D \vdash D}{C \vdash D, C \smallsetminus D} \qquad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}}{C \vdash D, \Upsilon} \quad \text{reduces to} \quad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}
$$

In words, if we use the hypothesis that C *excludes* D ($C \smallsetminus D$) to remove possible consequences of C of the form D from consideration, but the hypothesis $C \smallsetminus D$ was itself derived from C by an inference that yields the hypothesis D as a possible consequence, then nothing has been achieved by performing such a pair of operations. We conclude not only that the two derivations have the same deductive consequences but also that in some sense they may be regarded as *the same deductive process*. The latter assertion may be disputed, but the above argument is the core of a *proof theoretic justification* of the introduction and elimination pair for subtraction. Here we assume that the primary operational meaning is given by the *elimination* rule, by which some conclusions are "excluded from consideration"; the *introduction* rule is shown to be in *harmony* with it (in the sense of Dummett [12]). The choice of the elimination rule as primary is also supported by the fact that it is *invertible* while the introduction rule, in its general form, is not.

A similar procedure may give a justification for disjunction.

$$
\curlyvee\text{-}\mathit{intro}\;\dfrac{H \vdash \Gamma, C, D}{H \vdash \Gamma, C \curlyvee D} \qquad \curlyvee\text{-}\mathit{elim}\;\dfrac{H \vdash \Gamma, C \curlyvee D \qquad C \vdash \Delta \qquad D \vdash \Upsilon}{H \vdash \Gamma, \Delta, \Upsilon}
$$

We have the following reduction:

$$
\curlyvee\text{-E}\;\dfrac{\curlyvee\text{-I}\;\dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}}{H \vdash \Upsilon, C \curlyvee D} \qquad \begin{matrix} d_2 \\ C \vdash \Gamma \end{matrix} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Upsilon, \Gamma, \Delta}
$$

reduces to

$$
\mathit{subst}\;\dfrac{\mathit{subst}\;\dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix} \qquad \begin{matrix} d_2 \\ C \vdash \Gamma \end{matrix}}{H \vdash \Upsilon, \Gamma, D} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Upsilon, \Gamma, \Delta}
$$

Again let's suppose that d_2 and d_3 are simply *assumptions*. Then the reduction is as follows:

$$
\curlyvee\text{-E}\;\dfrac{\curlyvee\text{ I}\;\dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}}{H \vdash \Upsilon, C \curlyvee D} \qquad C \vdash C \qquad D \vdash D}{H \vdash \Upsilon, C, D} \quad \text{reduces to} \quad \begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}
$$

In words, the $\curlyvee$-*introduction* rule connects two possible conclusions C and D into one $C \curlyvee D$ and the $\curlyvee$-*elimination* rule uses the resulting connection to join into the same deductive context two separate contexts, one resulting from C and the other from D. But C and D were already in the same context to start with, thus the possibility for joining the context was already there in the preconditions of the $\curlyvee$-*introduction* rule. Thus the two derivations have not only the same deductive consequences but also they may be regarded as the same deductive method. It is completely clear that the two rules are in *harmony*. We have chosen the *introduction* rule as giving the primary meaning also considering that it is invertible, while the *elimination* in its general form is not. It may be possible to argue that the context-joining operation exhibited by the $\curlyvee$-*elimination* rule is defining the operational meaning of co-intuitionistic disjunction.

If we take the $\curlyvee$-*introduction* rule as defining the operational meaning of (multiplicative) disjunction, we can say that it exhibits a possibility of connection between two conclusions given by the fact of being in the same context. It may be objected that the meaning of multiplicative disjunction is already given by the multiple-conclusion context; so the interpretation is in some sense circular. The objection is sensible, but it may only show a feature of such *meaning-as-use* interpretations that does not make them irrelevant. Making an implicit possibility of connection explicit is precisely what the $\curlyvee$-*introduction* rule does.

In this paper we shall not flesh out the *meaning-as-use* interpretation in full detail. However our pragmatic interpretation contributes to a justificationist approach by providing an analysis of the contribution to meaning given by *elementary expressions* in virtue of their *illocutionary force*. This analysis allows us to extend the *meaning-as-use* interpretation beyond intuitionistic logic. It is because we interpret the elementary expressions of co-intuitionistic logic as expressing the illocutionary force of a *hypothesis* that we are allowed to give co-intuitionistic disjunction a hypothetical mood and to justify its logical properties, which are very different from those of the usual *assertive* intuitionistic disjunction. By regarding co-intuitionism as the logic of the justification of *hypotheses*, we can explain and justify the duality between intuitionism and co-intuitionism in terms of common sense reasoning, in so far as the notion of a hypothesis can be seen as dual to that of an assertion.

We focus on the proposal of a semantic for *multiplicative linear* bi-intuitionistic logic, a *pragmatic dialogue interpretation* of co-intuitionism and to bi-intuitionism in which the two views **H.1** and **H.2** are combined; such a dialogue interpretation uses the general notion of a method that is characteristic of *linear intuitionistic logic* but is applied here to transform not only proofs to proofs, but also *non-conclusive evidence* into *non-conclusive evidence*. In this framework we have a *stricter* interpretation for *linear* co-intuitionistic multiplicative disjunction than that of a "contextual compatibility" evoked above, which is implicit in the form of the co-intuitionistic consequence relation. The dialogue interpretation does not rely on a meta-theoretic understanding of the meaning of a multiplicative disjunction. Moreover such an interpretation can be formalized within *multiplicative intuitionistic linear logic with products*, in a way that evokes Chu's construction [3] (sketched in Appendix B in Part II).

3. Pragmatic interpretation of bi-intuitionism

We develop our interpretation by expanding and reinterpreting Dalla Pozza and Garola's *pragmatic interpretation of intuitionistic logic* [10], which is in accordance with M. Dummett's suggestion that intuitionism is the logic of *assertions* and of their justifications. The main feature of Dalla Pozza and Garola's approach is to take *elementary expressions* of the form $\vdash P$, where Frege's symbol "$\vdash$" represents an (impersonal) *illocutionary force of assertion* and P is a proposition. The language of Dalla Pozza and Garola $\mathcal{L}^P$ [10], extended with modal operators "$\Box$" and "$\Diamond$", is as follows. We are given an infinite sequence of propositional atoms $p_0, p_1, \ldots,$ *propositional constants* $\mathbf{t}$ and $\mathbf{f}$, denoting *true* and *false* propositions, respectively, and *sentential constants* $\curlyvee$ and $\mathbf{u}$, denoting an *always justified* assertion and an *always unjustified* assertion, respectively. The expressions A of the (modally extended) language $\mathcal{L}^P$ are given by the following grammar:

$$\begin{aligned} P, Q &:= p \mid \mathbf{t} \mid \mathbf{f} \mid \neg P \mid P \to Q \mid P \wedge Q \mid P \vee Q \mid \Box P \mid \Diamond P \\ A, B &:= \vdash P \mid \curlyvee \mid \mathbf{u} \mid A \supset B \mid A \cap B \mid A \cup B \end{aligned} \tag{3.1}$$

The intuitionistic fragment of Dalla Pozza and Garola $\mathcal{L}^P$ is obtained by restricting *elementary formulas* $\vdash P$ to contain only *propositional atoms* "p_i" in the scope of the sign of illocutionary force "$\vdash$". Thus intuitionistic elementary formulas denote assertions of propositions regarded as atoms. Intuitionistic negation "$\sim$" is defined as $\sim A = A \supset \mathbf{u}$.

The justification of intuitionistic formulas is given precisely by Brouwer-Heyting-Kolmogorov's interpretation of intuitionistic connectives: the justification of $\vdash p$ is given by *conclusive evidence* for p (e.g., a proof of the mathematical proposition p) and the justification of an *implication* $A \supset B$ is a method that transforms a justification of A into a justification of B. Moreover a justification of a conjunction $A \cap B$ is a pair $\langle j, k\rangle$ where j is a justification of A and k a justification of B; a justification of a disjunction $A_0 \cup A_1$ is a pair $\langle j, 0\rangle$ where j is a justification of A_0 or $\langle k, 1\rangle$ where k is a justification of A_1.

To be sure, from an intuitionistic viewpoint the proposition p must be such that conclusive evidence for it can be effectively given: the (informal) proof justifying $\vdash p$ must be intuitionistic. Obviously we cannot let p be $q \vee \neg q$ where q is intuitionistically undecidable and claim that $\vdash (p \vee \neg p)$ is justified by a classical proof. Thus Dalla Pozza and Garola assume that in the representation of intuitionism the proposition p must be regarded as atomic. If this is granted, then the expressions of $\mathcal{L}^P$ are *types of justification methods*; in a *propositions as types* framework they are *intuitionistic propositions*.

Having introduced the consideration of illocutionary forces in the elementary expression of logical languages, we can then ask in which sense intuitionistic types are *assertive expressions*: do molecular expressions inherit illocutionary force from their elementary components? Is an illocutionary assertive force implicit in the way of presenting their justification? This is an interesting question, which Dalla Pozza and Garola do not give an explicit answer to. It seems clear to us that the molecular expressions of the above language must have an *"assertive mood"*, which sets them apart from other forms of reasoning, say, in a hypothetical or conjectural mood.

Gödel, McKinsey, Tarski [20] and Kripke's modal **S4** interpretation is naturally considered here as a *reflection* of the *pragmatic layer* of the logic for pragmatics into the *semantic layer*, where the image $\Box A'$ of a pragmatic expression A is indeed a proposition of classical modal logic **S4**, and the necessity operator of **S4** is read as an operator of "abstract knowability". Briefly put, the modal meaning of pragmatic assertions is provided by a translation of pragmatic connectives where

$$\begin{array}{rlrl} (\vdash p)^M = & \Box p, & (A \supset B)^M & = \Box(A^M \to B^M), \\ (\curlyvee)^M = & \mathbf{t}, & (\mathbf{u})^M & = \mathbf{f} \\ (A \cap B)^M = & A^M \wedge B^M, & (A \cup B)^M & = A^M \vee B^M. \end{array} \tag{3.2}$$

Thus Dalla Pozza and Garola develop a two-layers formal system where the propositions p occurring in an elementary expression $\vdash p$ are interpreted according to classical semantics. Moreover they seem to think that the meaning of the *intensonal* expressions of intuitionistic pragmatics are adequately represented by their *extensional* translations in **S4**. Finally they develop their pragmatic interpretation in a *classical metatheory*. Thus what they obtain is a constructive interpretation of intutionism in a classical framework. This is certainly unacceptable to an intuitionistic philosopher but is fully in the spirit of Dalla Pozza and Garola's pragmatics: broadly speaking, their goal is to show how classical logic, as a theory of truth, can be reconciled with intuitionism, as a theory of justified assertability, by the principle that a *"change of logic is a change of subject matter"*.

We believe that such a classical twist in not essential to the project of an intuitionistic pragmatics and indeed that not much needs to be changed to obtain a *bona fide* representation of intuitionism. Granted that the *"semantic projection"* into **S4** is only an *"extensional abstract interpretation"* of intuitionistic pragmatic expressions and that we must work in an *intuitionistic metatheory*, the *pragmatic interpretation of intuitionistic logic* becomes compatible with intuitionistically acceptable interpretations according to a *justificationist* approach, either in a theory of *meaning-as-use* or in some kind of *game-theoretic semantics*.

3.1. Co-Intuitionistic Logic as a logic of hypotheses

A clear example of how the essential properties of a logic depend on the epistemic attitudes expressed in elementary formulas is given by assigning the illocutionary force of *hypothesis*, rather than of *assertion*, to elementary formulas of co-intuitionistic logic. When molecular co-intuitionistic formulas acquire a *hypothetical mood*, the meaning of connectives changes: assuming that we know what counts as a justification of an *elementary hypothesis*, the meaning of *hypothetical disjunction* $C \curlyvee D$ and *hypothetical conjunction* $C \curlywedge D$ are obviously different from their assertive counterparts. Lack of justification for the hypothesis C seems enough justification for doubting that C, and conversely; thus the principle $C \curlyvee \frown C$ is valid. If C is justifiably given the illocutionary force of a hypothesis, rather than of an assertion, we cannot exclude that there may be justified reasons to set aside such a hypothesis, i.e., that we may *justifably* entertain a *doubt* ($\frown C$) about C; the converse also holds. Thus no contradiction derives from a simultaneous considerations $C \curlywedge \frown C$ of the hypotheses C and $\frown C$. Notice however that it is only because of the *hypothetical mood* of subtraction that the law of excluded middle is valid and para-consistency holds beyond dispute: since $\frown C$ is definable as $\mathbf{j} \smallsetminus C$, a non-hypothetical reading of $\frown C$ as *"the valid hypothesis excludes C"* may make the law of excluded middle intuitionistically problematic for co-intuitionistic disjunction.

It is even possible to have *mixed connectives* operating on assertive and hypothetical sentences and building assertive or hypothetical connectives in the framework of *polarized bi-intuitionistic* logic: this has been done in [5] and completeness of the resulting logic with respect to the classical **S4** translation has been checked. Here we consider only the fragment of such logic that allows us to express the *duality* between intuitionism and co-intuitionism.

Our *co-intuitionistic logic of hypothesis* is built from elementary hypothetical expressions $\mathcal{H}\, p$ sentential constants "$\curlywedge$" for a hypothesis which is *always unjustified* and "$\mathbf{j}$" for an *always justified hypothesis*, using the connectives *subtraction* $C \smallsetminus D$ (*"C excludes D"*), *hypothetical disjunction* $C \curlyvee D$ and *hypothetical conjunction* $C \curlywedge D$.

$$C, D \;:=\; \mathcal{H}\, p \mid \;\curlywedge\; \mid \mathbf{j} \mid C \smallsetminus D \mid C \curlyvee D \mid C \curlywedge D \tag{3.3}$$

Finally, *supplement* (*weak negation*) "$\frown$" is defined as $\frown C = (\mathbf{j} \smallsetminus C)$.

A straightforward extension of the **S4** modal translation to co-intuitionism is as follows:

$$\begin{array}{rll rl} (\mathcal{H}\, p)^M = & \Diamond p, & & (C \smallsetminus D)^M & = \Diamond(C^M \wedge \neg D^M), \\ (\curlywedge)^M = & \mathbf{f}, & & (\mathbf{j})^M & = \mathbf{t} \\ (C \curlyvee D)^M = & C^M \vee D^M, & & (C \curlywedge D)^M & = C^M \wedge D^M, \end{array} \tag{3.4}$$

where "$\neg$", "$\wedge$", "$\vee$" are the classical connectives, $\mathbf{t}$ and $\mathbf{f}$ the truth values. Here we clearly see that such an extension of Gödel's, McKinsey and Tarski's and Kripke's translation unacceptably collapses assertive and hypothetical constants:

$$(\curlyvee)^M \;=\; \mathbf{t} \;=\; (\mathbf{j})^M \quad \text{and} \quad (\curlywedge)^M \;=\; \mathbf{f} \;=\; (\mathbf{u})^M. \tag{3.5}$$

But what constitutes a *justification* for a *hypothesis* ($\mathcal{H}\, p$) and how does it differ from a justification of an *assertion* ($\vdash p$)? In the familiar Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic evidence for a mathematical statement p is a proof of it; in the case of non-mathematical assertive statements, we speak of *conclusive evidence* for p. What constitutes *conclusive* or *inconclusive evidence* for p depends on the context and scientific discipline.

Consider for example, the *theory of argumentation* in legal reasoning. Here six *proof-standards* have been identified from an analysis of legal practice: *no evidence at alll*, *scintilla of evidence*, *preponderance of evidence*, *clear and convincing evidence*, *beyond reasonable doubt* and *dialectical validity*, in a linear order of strength [15, 7]. Can such distinctions be taken up in our approach in some way? It seems that a *scintilla of evidence* suffices to justify $\mathcal{H}\, p$, making the hypothesis that p, and that *dialectical validity* ought to coincide with assertability $\vdash p$, which in our framework is *conclusive evidence.* The other proof-standards are defined through probabilities; this goes beyond our purely logical approach here.

If we assume the notion of "negative evidence" (*evidence against* the truth of a proposition) as basic, in addition to "positive evidence" (*evidence for*), then another logical relation is evident between *scintilla of evidence* and *conclusive evidence*, in addition to the order of strength: *we cannot have at the same time* conclusive evidence for *and a* scintilla of evidence against *the truth of a proposition.* On this basis we can attempt an interpretation of intuitionistic and co-intuitionistic connectives which is reminiscent of *game semantics* and also of Nelson's treatment of *constructive falsity* (see [23]).

4. The language and sequent calculus of "polarized" bi-intuitionism

We consider two formal systems for *"polarized" bi-intuitionism* as our *"official languages"*. One, the logic **AH** of *assertions and hypotheses*, is a conservative extension of both intuitionistic and co-intuitionistic logic; the other **MLAH** is the multiplicative fragment of the linear version of **AH**.

Given an infinite sequence p_0, p_1 ... of propositional letters, the language of **AH** consists of two sides:

- the *assertive intuitionistic side*, built from elementary expressions of the form "$\vdash p$" (*elementary assertions*) and from the sentential constants $\curlyvee$ (*assertive validity*), $\mathbf{u}$ (*assertive absurdity*), using implication ($\supset$), conjunction ($\cap$) and disjunction ($\cup$);

- the *hypothetical co-intuitionistic side* built from *elementary hypotheses* "$\mathcal{H}\, p$" and the sentential constants $\curlywedge$ (*hypothetical absurdity*) and $\mathbf{j}$ (*hypothetical validity*) using subtraction ($C \smallsetminus D$), disjunction ($\curlyvee$) and conjunction ($\curlywedge$).

Intuitionistic negation $\sim_u A$ and co-intuitionistic *supplement* ${}_j\frown C$ are defined:

$$\sim_u A = A \supset \mathbf{u} \qquad {}_j\frown C = \mathbf{j} \smallsetminus C.$$

Two negations relate the two sides, a *strong* one $(\ )^{h\perp a}$ transforming a hypothesis C into an assertion $(C)^{h\perp a}$ and a *weak* one $(\ )^{a\perp h}$ transforming an assertion A into a hypothesis $(A)^{a\perp h}$. Through these negations the duality between the intuitionistic and the co-intuitionistic sides is expressed within the language.

Definition 4.1 (intuitionistic assertions, co-intuitionistic hypotheses).

- assertive intuitionistic formulas:
  $A, B \ := \ \vdash p \mid \ \curlyvee \ \mid \mathbf{u} \mid A \supset B \mid A \cap B \mid A \cup B \mid (C)^{h\perp a}$
- hypothetical co-intuitionistic formulas:
  $C, D \ := \ \mathcal{H}\, p \mid \ \curlywedge \ \mid \mathbf{j} \mid C \smallsetminus D \mid C \curlyvee D \mid C \curlywedge D \mid (A)^{a\perp h}$
- defined negations:
  $\sim_u A \ := \ A \supset \mathbf{u} \qquad {}_j\frown C \ := \ \mathbf{j} \smallsetminus C.$

In this paper we shall not consider assertive disjunction $(A \cup B)$ *and hypothetical conjunction* $C \curlywedge D$.

Remark 4.1. Since the two negations are distinguished unambiguously from the context, we shall usually drop the subscripts from $(\ )^{a\perp h}$, $(\ )^{h\perp a}$ and write simply $(\ )^{\perp}$ for both. However it is essential to remember that we always have two distinct operations and that they should not be confused with the orthogonality operator $(\ )^{\perp}$ of linear logic.[5]

Proposition 4.1. *Extend the* **S4** *translation given by* (3.2) *and* (3.4) *with the following conditions*:

$$\left((A)^{a\perp h}\right)^M \ = \ \neg A^M \qquad \left((C)^{h\perp a}\right)^M \ = \ \neg C^M \tag{4.1}$$

Then in the **S4** *translation we have* $(A)^M \equiv \Box A^M$ *and* $(C)^M \equiv \Diamond C^M$.

[5] In [5, 2] the same symbol "$\sim$" was used for (definable) intuitionistic negation $\sim_u A$ and for the duality $(C)^{h\perp a}$. Also the same symbol "$\frown$" was used for (definable) co-intuitionistic supplement ${}_j\frown C$ and for the duality $(A)^{a\perp h}$. Mathematically, it was an unfortunate choice since dualities and defined negations have totally different properties.

Note. An equivalent translation would be obtained if we translated the dualities as negations, letting $\left((A)^{a\perp h}\right)^M = \Diamond\neg A^M$ and $\left((C)^{h\perp a}\right)^M = \Box\neg C^M$ as in [5, 2]. Indeed in classical **S4** by Proposition 4.1 we have $\Diamond\neg A^M \equiv \neg\Box A^M \equiv \neg A^M$ and $\Box\neg C^M \equiv \neg\Diamond C^M \equiv \neg C^M$.

The main distinguishing feature of the logic **AH** from Rauszer's bi-intuitionism is given by the following equivalences, which are obviously preserved by the **S4** translation:

$$(A)^{\perp\perp} \equiv A \qquad (C)^{\perp\perp} \equiv C. \tag{4.2}$$

4.1. Informal Interpretation

The language of polarized bi-intuitionism has an informal "intended interpretation" where formulas denote *types of acts of assertion and of hypothesis* and must be given *justification conditions*, namely, epistemic conditions that constitute *evidence for* illocutionary acts of these types. We take the notions of *"conclusive evidence"* and *"scintilla of evidence"* as primitive notions, with the obvious ordering, namely, we assume that conclusive evidence is also a scintilla of evidence, but not conversely. Thus we can define simultaneously what it means for assertive and hypothetical expressions to be justified.

DEFINITION 4.2. (**a**.i) The assertion $\vdash p$ is justified by *conclusive evidence* of the truth of p;
(**a**.ii) the assertion Υ is always justified and assertion **u** is never justified;
(**a**.iii) $A \supset B$ is justified by a method transforming conclusive evidence for A into conclusive evidence for B; evidence for $C^\perp$ is a method transforming evidence for C into a contradiction;
(**a**.iv) $A \cap B$ is justified by conclusive evidence for both A and B; $A \cup B$ is justified by conclusive evidence either for A or for B.

Dually:
(**h**.i) the hypothesis $\mathcal{H}\, p$ is justified by *a scintilla of evidence* of the truth of p;
(**h**.ii) the hypothesis ⅄ is never justified and hypothesis **j** is always justified;
(**h**.iii) $C \smallsetminus D$ is justified by a scintilla of evidence for C together with a method showing that evidence for C and for D are incompatible; evidence for $A^\perp$ is a justification for disregarding evidence for A, i..e, for *doubting* of justifications of A;

($\mathbf{h}$.*iv*) $C \curlyvee D$ is justified by a scintilla of evidence for C or for D; $C \curlywedge D$ is justified by a scintilla of evidence for both C and D.

Remark 4.2. (i) We have four *justification values*, assertive validity ($\curlyvee$) and invalidity ($\mathbf{u}$) and hypothetical invalidity ($\curlywedge$) and validity ($\mathbf{j}$). We cannot identify assertive and hypothetical validity, nor hypothetical and assertive invalidity; we must think of $\mathbf{u}$ as an expression $\vdash p$ which is always invalid although p may be sometimes true, and similarly $\mathbf{j}$ as an expression $\mathcal{H}\, p$ which is always valid although p may be sometimes false.

(ii) Assertive validity $\curlyvee$ and hypothetical invalidity $\curlywedge$ can be related to $\top$ and $\mathbf{0}$ of linear logic as they are interpreted categorically as the terminal and the initial object in their respective categories. However there are no obvious reasons for relating $\mathbf{u}$ with $\perp$ and $\mathbf{j}$ with $\mathbf{1}$.

(iii) The meaning of subtraction is a delicate point. The accepted informal interpretation of $A \smallsetminus B$ as "A excludes B" was proposed by I. Urbas [32], in place of "A but not B", as suggested by N. Goodman [14].[6] Here however "$C \smallsetminus D$" has a hypothetical mood. Suppose we have a method showing incompatibility between any evidence for C and any evidence for D, the hypothetical character of subtraction may come either (a) from the fact that the actual evidence for C may be fairly weak or (b) from the nature of the evidence for incompatibility. In this former case the hypothetical mood for $_{\mathbf{j}}\frown C$ or $_{\mathbf{j}}\frown A$ would depend on the fact that evidence for $\mathbf{j}$ is weak. No such hypothesis is necessary in the latter case.

4.2. Sequent calculus for Polarized Bi-intuitionism

The *sequent calculus* **AH-G1** has sequents of one of the forms

$$\Theta\ ;\ \Rightarrow\ A\ ;\ \Upsilon \quad \text{or} \quad \Theta\ ;\ C\ \Rightarrow\ ;\ \Upsilon$$

where the multiset Θ and A are *assertive* formulas and the multiset Υ and C are hypothetical formulas. We use the abbreviation

$$\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon$$

where *exactly one* of ϵ or ϵ' is non-null. The inference rules of **AH-G1** are in the following tables 4.1, 4.2, 4.3, 4.4, 4.5.

[6] We thank the anonymous referee for this reference.

logical axiom:

$$A\ ;\ \Rightarrow\ A\ ;$$

cut$_1$:

$$\frac{\Theta\ ;\ \Rightarrow\ A\ ;\ \Upsilon \qquad A, \Theta'\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon'}{\Theta, \Theta'\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \Upsilon'}$$

logical axiom:

$$;\ C\ \Rightarrow\ ;\ C$$

cut$_2$:

$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, C \qquad \Theta'\ ;\ C\ \Rightarrow\ \Upsilon'}{\Theta, \Theta'\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \Upsilon'}$$

Proper axioms of assertions and hypotheses

$\vdash / \mathcal{H}$ *left*	$\vdash /\mathcal{H}$ *right*
$\vdash p\ ;\ \mathbf{j}\ \Rightarrow\ ;\ \mathcal{H}\, p$	$\vdash p\ ;\ \Rightarrow\ \mathbf{u}\ ;\ \mathcal{H}\, p$

Table 4.1. Identity rules

Remark 4.3. (i) Sequents with *focus*, as in the linear logic literature, single out formulas that cannot be conclusions of *weakening* or *contraction* and thus implement an essential feature of Gentzen's intuitionistic restriction to a single-formula in the succedent (and similarly the dual co-intuitionistic restriction).

(ii) The presence of two focusses and the alternation between them in our system is best understood from the *duality rules* 4.5. By the equivalence 4.2, using the ${}_c\bot_a$-*left* rule, a sequent $\Gamma; \Rightarrow A; \Delta$ is equivalent to $\Delta^\bot, \Gamma; \Rightarrow A;$ (where if $\Delta = D_1, \ldots, D_n$ then $\Delta^\bot = D_1^\bot, \ldots, D_n^\bot$). In a categorical model the latter sequent corresponds to a morphism in the "intuitionistic category"; essentially, we are building an intuitionistic proof. Similarly, using the ${}_a\bot_h$ rule a sequent $\Gamma; C \Rightarrow; \Delta$ is equivalent to $; C \Rightarrow; \Delta, \Gamma^\bot$, corresponding to an arrow in a "co-intuitionistic category".

On the contrary, the ${}_c\bot_a$-*right* rule sends us from a co-intuitionistic proof to an intuitionistic one, thus represents the action of the contravariant duality functor from the co-intuitionistic category to the intuitionistic one. Dually, the ${}_a\bot_c$-*right* rule represents the contravariant duality functor from the intuitionistic category to the o-intuitionistic one.

contraction left $$\frac{A, A, \Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{A, \Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}$$	*contraction right* $$\frac{\Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, C, C}{\Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, C}$$
weakening left $$\frac{\Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{A, \Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}$$	*weakening right* $$\frac{\Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{\Theta \,; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, C}$$

Table 4.2. **AH-G1** structural rules

(iii) The core fragment of *intuitionistic logic* consists of the rules for assertive conjunction ($\cap$), implication ($\supset$) and validity ($\curlyvee$), without the rules for assertive disjunction ($\cup$) and absurdity ($\mathbf{u}$); in this core fragment the symbol "$\mathbf{u}$" in the definition of intuitionistic negation is just a sentential constant without special properties. Dually, the core fragment of *co-intuitionistic logic* has the rules for assertive disjunction ($\curlyvee$), subtraction ($\smallsetminus$) and absurdity ($\curlywedge$), without the rules for hypothetical conjunction ($\curlywedge$) and validity ($\mathbf{j}$), which in the definition of co-intuitionistic negation is just a sentential constant. *We shall consider only the core fragment of intutionistic and co-intuitionistic logic.*

(iv) The form of bi-intuitionistic sequents, where only one expression occurs in the focusing area, forces the rules for assertive disjunction and hypothetical conjunction to have *additive form*; thus "$\cup$" has the disjunction property and $\nvdash A \cup \sim A$; dually, $\curlywedge$ has the *conjunction property* ($C \curlywedge D \vdash$ implies $C \vdash$ or $D \vdash$) and is para-consistent ($C \curlywedge \frown C \nvdash \curlyvee$).

(v) On the other hand, the rules for assertive conjunction and hypothetical disjunction could be given in the *additive* or in the *multiplicative* form; in presence of the structural rules of the structural rules of weakening and contraction the two formulations are equivalent. For the categorical considerations sketched above, we give *additive* rules for assertive disjunction "$\cap$" and *multiplicative* rules for hypothetical disjunction "$\curlyvee$".

assertive validity axiom:

$$\Theta \;;\; \Rightarrow\; \curlyvee \;;\; \Upsilon$$

$\supset$ *right:*

$$\frac{\Theta, A_1 \;;\; \Rightarrow\; A_2 \;;\; \Upsilon}{\Theta \;;\; \Rightarrow\; A_1 \supset A_2 \;;\; \Upsilon}$$

$\supset$ *left :*

$$\frac{\Theta_1 ; \;\Rightarrow\; A_1 \;;\; \Upsilon_1 \qquad A_2, \Theta_2 \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon_2}{A_1 \supset A_2, \Theta_1, \Theta_2 \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon_1, \Upsilon_2}$$

$\cap$ *right:*

$$\frac{\Theta \;;\; \Rightarrow\; A_1 \;;\; \Upsilon \qquad \Theta \;;\; \Rightarrow\; A_2 \;;\; \Upsilon}{\Theta \;;\; \Rightarrow\; A_1 \cap A_2 \;;\; \Upsilon}$$

$\cap$ *left:*

$$\frac{A_i, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{A_0 \cap A_1, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}$$

for $i = 0, 1$.

assertive absurdity axiom:

$$\mathbf{u}, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon$$

assertive disjunction left

$$\frac{A, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon \qquad B, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{A \cup B, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}$$

assertive disjunction right
(two rules)

$$\frac{\Theta \;;\; \Rightarrow\; A_i \;;\; \Upsilon}{\Theta \;;\; \Rightarrow\; A_0 \cup A_1 \;;\; \Upsilon}$$

for $i = 0,\ 1$

Table 4.3. **AH-G1** intuitionistic rules

Using standard techniques (see [5]) one can prove the following proposition.

PROPOSITION 4.2. *The sequent calculus* **AH-G1** *is sound and complete for the Kripke semantics over preordered frames determined the* **S4** *interpretation in* (3.2), (3.4) *and* (4.1). *The rules for cut are admissible in* **AH-G1**.

Let us use the following abbreviations:

$$\boxdot C \;:=\; \sim_u (C^\perp) \quad \text{and} \quad \diamonddot A \;:=\; {}_j\frown (A^\perp)$$

PROPOSITION 4.3. *The following sequents are provable in* **AH-G1**:

(i) $A^{\perp\perp} \;;\; \Rightarrow A$; *and* $A \;; \Rightarrow\; A^{\perp\perp}$;
and dually $;C \Rightarrow\; ;C^{\perp\perp}$ *and* $;C^{\perp\perp} \Rightarrow\; ;C$.
(ii) $A \;;\; \Rightarrow\; \boxdot\, \diamonddot A$; *and* $;\; \diamonddot \boxdot C \;\Rightarrow\; ;\; C$.
(iii) $M \supset\; \boxdot C \;\Rightarrow\; \boxdot(M^\perp \curlyvee C)$ *and* $\boxdot(M^\perp \curlyvee C) \;; \Rightarrow\; ;\; M \supset\; \boxdot C$.

PROOF. (ii) and (iii)

hypothetical absurdity axiom:

$$\Theta \,;\, \curlywedge \;\Rightarrow\; ;\, \Upsilon$$

$\smallsetminus$ *right:*

$$\frac{\Theta_1 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon_1, C_1 \qquad \Theta_2 \,;\, C_2 \;\Rightarrow\; ;\, \Upsilon_2}{\Theta_1, \Theta_2 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon_1, \Upsilon_2, C_1 \smallsetminus C_2}$$

$\smallsetminus$ *left:*

$$\frac{\Theta;\, C_1 \;\Rightarrow\; ;\, \Upsilon, C_2}{\Theta \,;\, C_1 \smallsetminus C_2 \;\Rightarrow\; ;\, \Upsilon}$$

$\curlyvee$ *right:*

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C_0, C_1}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C_0 \curlyvee C_1}$$

$\curlyvee$ *left:*

$$\frac{\Theta_1 \,;\, C_1 \;\Rightarrow\; ;\, \Upsilon_1 \qquad \Theta_2 \,;\, C_2 \;\Rightarrow\; ;\, \Upsilon_2}{\Theta_1, \Theta_2 \,;\, C_1 \curlyvee C_2 \;\Rightarrow\; ;\, \Upsilon_1, \Upsilon_2}$$

hypothetical validity axiom:

$$\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \mathbf{j}$$

hypothetical conjunction left
(two rules)

$$\frac{\Theta \,;\, C_i \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, C_0 \curlywedge C_1 \;\Rightarrow\; ;\, \Upsilon}$$

for $i = 0, 1$

hypothetical conjunction right

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C \qquad \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, D}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C \curlywedge D}$$

Table 4.4. **AH-G1** co-intuitionistic rules

$$\dfrac{\dfrac{\dfrac{; \Rightarrow \mathbf{u} \,;\, \mathbf{j} \qquad \dfrac{A \,;\, \Rightarrow\; A \,;}{A \,;\; A^{\perp} \Rightarrow\; ;}\;{}_{a}\!\perp_{h}\!L}{A \,;\; \Rightarrow\; \mathbf{u} \,;\; \underbrace{{}_{j}\frown(A^{\perp})}_{\diamonddot\, A}}\;\smallsetminus R}{A, (\diamonddot\, A)^{\perp} \,;\; \Rightarrow\; \mathbf{u} \,;}\;{}_{h}\!\perp_{a}\!L}{A \,;\; \Rightarrow\; \underbrace{\sim_{u}((\diamonddot\, A)^{\perp})}_{\boxdot\diamonddot A};}\;\supset R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{; C \;\Rightarrow\; ; C}{\Rightarrow\; C^{\perp} \,;\, C}\;{}_{h}\!\perp_{a}\!R \qquad \mathbf{u} \,;\, \mathbf{j} \;\Rightarrow\; ;}{\underbrace{\sim_{u}(C^{\perp})}_{\boxdot\; C} \,;\; \mathbf{j} \;\Rightarrow\; ;\; C}\;\supset L}{;\; \mathbf{j} \;\Rightarrow\; ;\; (\boxdot\, C)^{\perp}, C}\;{}_{a}\!\perp_{h}\!R}{\underbrace{{}_{j}\frown((\boxdot\, C)^{\perp})}_{\diamonddot\boxdot C} \,;\; \Rightarrow\; ;\; C}\;\smallsetminus L$$

${}_h{\perp}_a$ *right:* $\dfrac{\Theta\,;\,C\ \Rightarrow\ ;\,\Upsilon}{\Theta\,;\ \Rightarrow\ C^{\perp}\,;\,\Upsilon}$	${}_h{\perp}_a$ *left:* $\dfrac{\Theta;\,\epsilon\ \Rightarrow\ \epsilon'\,;\,\Upsilon,C}{C^{\perp},\Theta\,;\,\epsilon\ \Rightarrow\ \epsilon'\,;\,\Upsilon}$
${}_a{\perp}_h$ *right:* $\dfrac{\Theta,A\,;\,\epsilon\ \Rightarrow\ \epsilon'\,;\,\Upsilon}{\Theta\,;\,\epsilon\ \Rightarrow\ \epsilon'\,;\,\Upsilon,A^{\perp}}$	${}_a{\perp}_h$ *left:* $\dfrac{\Theta;\ \Rightarrow\ A\,;\,\Upsilon}{\Theta\,;\,A^{\perp}\ \Rightarrow\ ;\,\Upsilon}$
u/j *left* $\mathbf{u}\,;\,\mathbf{j}\ \Rightarrow\ ;$	**u/j** *right* $;\ \Rightarrow\ \mathbf{u}\,;\,\mathbf{j}$

Table 4.5. **AH-G1** duality rules

$$
{\supset}R\;\dfrac{{}_h{\perp}_aL\;\dfrac{\curlyvee R\;\dfrac{{}_a{\perp}_hR\;\dfrac{{\supset}L\;\dfrac{M;\Rightarrow M\,; \qquad {\supset}L\;\dfrac{{}_h{\perp}_aR\;\dfrac{;\,C\ \Rightarrow\ ;C}{;\ \Rightarrow\ C^{\perp}\,;\,C} \qquad \mathbf{u}\,;\Rightarrow\mathbf{u}\,;}{\boxdot C\,;\ \Rightarrow\ \mathbf{u}\,;\,C}}{M,M\supset\boxdot C\,;\ \Rightarrow\ \mathbf{u}\,;\,C}}{M\supset\boxdot C\,;\ \Rightarrow\ \mathbf{u}\,;\,M^{\perp},C}}{M\supset\boxdot C\,;\ \Rightarrow\ \mathbf{u}\,;\,M^{\perp}\curlyvee C}}{M\supset\boxdot C,(M^{\perp}\curlyvee C)^{\perp}\,;\ \Rightarrow\ \mathbf{u}\,;}}{M\supset\boxdot C\,;\ \Rightarrow\ \boxdot(M^{\perp}\curlyvee C)\,;}
$$

$$
{\supset}R\;\dfrac{{\supset}R\;\dfrac{{}_h{\perp}_aL\;\dfrac{{\supset}L\;\dfrac{{}_h{\perp}_aR\;\dfrac{\curlyvee L\;\dfrac{{}_a{\perp}_hL\;\dfrac{M;\Rightarrow M;}{M\,;\,M^{\perp}\ \Rightarrow\ ;} \qquad ;\,C\ \Rightarrow\ ;C}{M\,;\,M^{\perp}\curlyvee C\ \Rightarrow\ ;\,C}}{M\,;\ \Rightarrow(M^{\perp}\curlyvee C)^{\perp}\,;\,C} \qquad \mathbf{u}\,;\ \Rightarrow\ \mathbf{u};}{\sim_u(M^{\perp}\curlyvee C)^{\perp},M\,;\ \Rightarrow\mathbf{u}\,;\,C}}{\boxdot(M^{\perp}\curlyvee C),M,C^{\perp}\,;\ \Rightarrow\mathbf{u};}}{\boxdot(M^{\perp}\curlyvee C),M\,;\ \Rightarrow\boxdot C;}}{\boxdot(M^{\perp}\curlyvee C)\,;\ \Rightarrow M\supset\boxdot C;}
$$

$\dashv$

The categorical definition of *dialogue chirality* is given in appendixes, where we also sketch the proof of the following proposition.

PROPOSITION 4.4. *Polarized bi-intuitionism forms a dialogue chirality.*

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A. Appendix I

A.1. Categorical models of bi-intuitionism

Definition A.1. A categorical model of **MLA** is built on a symmetric monoidal closed category. A *symmetric monoidal category* is a category $\mathcal{A}$ equipped with a bifunctor $\otimes\colon \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ and an object $\mathbf{1}$ (the identity of $\otimes$) together with natural isomorphisms

1. $\alpha_{A,B,C}\colon A\otimes(B\otimes C) \xrightarrow{\sim} (A\otimes B)\otimes C$;
2. $\lambda_A\colon 1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A\colon A\otimes 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B}\colon A\otimes B \xrightarrow{\sim} B\otimes A$.

which satisfy the following coherence diagrams.

$$\begin{tikzcd}
A\otimes(B\otimes(C\otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A\otimes\alpha_{B,C,D}"'] & (A\otimes B)\otimes(C\otimes D) \arrow[r, "\alpha_{A\otimes B,C,D}"] & (((A\otimes B)\otimes C)\otimes D \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A\otimes(B\otimes C))\otimes D \arrow[u, "\alpha_{A,B,C}\otimes id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B}\otimes id_C"'] & A\otimes(B\otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B\otimes A)\otimes C \arrow[r, "\alpha_{B,A,C}"'] & B\otimes(A\otimes C) \arrow[r, "id_B\otimes\gamma_{A,C}"'] & B\otimes(C\otimes A)
\end{tikzcd}$$

$$
\begin{tikzcd}
A \otimes (1 \otimes B) \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "id_A \otimes \lambda_B"'] & & (A \otimes 1) \otimes B \arrow[dl, "\rho_A \otimes id_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, "id_{A \otimes B}"'] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
& A \otimes B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes 1 \arrow[rr, "\gamma_{A,1}"] \arrow[dr, "\rho_a"'] & & 1 \otimes A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}
$$

The following equality is also required to hold: $\lambda_1 = \rho_1 \colon \bot \bullet \bot \to 1$.

Definition A.2. A *symmetric monoidal closed category* is a symmetric monoidal category $(\mathcal{A}, \otimes, 1, \alpha, \lambda, \rho, \gamma)$ such that for every object B of $\mathcal{A}$ the functor $\_ \otimes B \colon \mathcal{A} \to \mathcal{A}$ has a right adjoint $B \multimap \_ \colon \mathcal{A} \to \mathcal{A}$. Thus for every $A, C \in \mathcal{A}$ there is an object $B \multimap C$ and a natural bijection

$$\mathcal{A}(A \otimes B, C) \to (A, B \multimap C).$$

The *exponent* of B and C is an object $B \multimap C$ together with an arrow $\in_{B,C} \colon (B \multimap C) \otimes C \to C$ such that for any arrow $f \colon A \otimes B \to C$ there exists a unique $f^* \colon A \to (B \multimap C)$ making the following diagram commute:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "f"] \arrow[d, "f^* \otimes id_B"'] & C \\
(B \multimap C) \otimes B \arrow[ur, "\in_{B,C}"'] &
\end{tikzcd}
$$

In particular a *cartesian closed category* (with finite products) is a symmetric monoidal closed category where the *categorical product* $\times$ is the monoidal functor $\otimes$. A main example of cartesian closed category is **Set**, where product is the ordinary Cartesian product and exponents are defined from sets of functions.

Definition A.3. A categorical model of **MLH** is a symmetric monoidal category $(\mathcal{H}, \wp, \bot, \alpha, \lambda, \rho, \gamma)$, such that for every $D \in \mathcal{H}$ the functor $(D \wp \_) \colon \mathcal{H} \to \mathcal{H}$ has a *left* adjoint $(\_ - D) \colon \mathcal{H} \to \mathcal{H}$. Thus for every $C, E \in \mathcal{H}$ there is an object $C - D$ and a natural bijection

$$\mathcal{H}(C, D \wp E) \to (C - D, E).$$

The co-exponent of C and D is an object $C - D$ together with an arrow $\ni_{D,C}\colon C \to (C-D) \wp D$ such that for any arrow $f\colon C \to D \wp E$ there exists a unique $f_*\colon (C-D) \to E$ making the following diagram commute:

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\ni_{D,C}"'] & & E \wp D \\
& (C-D) \wp D \arrow[ur, "f_* \wp id_D"'] &
\end{tikzcd}$$

Lemma (Crolard [8]). *In the category* **Set** *the co-exponent $C - D$ of two sets C and D is defined if and only if $C = \emptyset$ or $D = \emptyset$.*

Proof. In **Set** coproducts are *disjoint unions*, write $E \oplus D$ for the disjoint union of E and D. If $C \neq \emptyset \neq D$ then the functions f and $\ni_{D,C}$ for every $c \in C$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \oplus 1_D$ leaves the side unchanged. Hence, if we take a nonempty set E and f with the property that for some c different sides are chosen by f and $\ni_{D,C}$, then the diagram does not commute. It is clear that such a failure does occur in *any category* where coproducts involves a choice between the arguments: in logic this is the case of an *additive* disjunction such as the intuitionistic one $(C \cup D)$ or the linear *plus* $(C \oplus D)$. $\dashv$

A.2. Dialogue chiralities

The concept of *chirality* (see Melliès [22]) is useful to study a pair of structures $(\mathcal{A}, \mathcal{B})$, where one of the two structures cannot be defined simply as the opposite of the other and the duality has to be somehow "relaxed". The case of models of bi-intuitionism is to the point: here we have two monoidal categories, where the "intuitionistic" structure $\mathcal{A}$ is cartesian closed, but by Crolard's theorem the "co-intuitionistic" one cannot be just $\mathcal{A}^{op}$.

Definition A.4. A dialogue chirality on the left is a pair of monoidal categories $(\mathcal{A}, \wedge, \mathrm{true})$ and $(\mathcal{B}, \vee, \mathrm{false})$ equipped with an adjunction

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"]
\end{tikzcd}$$

whose unit and counit are denoted as

$$\eta\colon Id \quad\to\quad R\circ L \qquad \epsilon\colon L\circ R\to Id$$

together with a monoidal functor[7]

$$(-)^* \quad ; \quad \mathcal{A} \quad\to\quad \mathcal{B}^{op(0,1)}$$

and a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle m\wedge a|b\rangle \quad\to\quad \langle a|m^*\vee b\rangle$$

natural in m, a, b *(curryfication)*. Here the bracket $\langle a|b\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathcal{A}$:

$$\langle a|b\rangle \quad = \quad \mathcal{A}(a, R(b)).$$

The family χ is moreover required to make the diagram

$$\begin{tikzcd}
\langle (m\wedge n)\wedge a \mid b\rangle \arrow[rr, "\chi_{m\wedge n}"'] \arrow[d, "assoc."'] & & \langle a \mid (m\wedge n)^*\vee b\rangle \\
 & = & \\
\langle m\wedge (n\wedge a) \mid b\rangle \arrow[r, "\chi_m"] & \langle n\wedge a \mid m^*\vee b\rangle \arrow[r, "\chi_n"] & \langle a \mid n^*\vee (m^*\vee b)\rangle \arrow[uu, "assoc.\ monoid.\ of\ (-)^*"']
\end{tikzcd}$$

commute for all objects a, m, n, and all morphisms $f\colon m\to n$ of the category $\mathcal{A}$ and all objects b of the category $\mathcal{B}$.

We sketch the construction of a chirality "from the syntax" of the bi-intuitionistic calculus. Let $(\mathcal{A}, \wedge, \mathbf{true})$ be the monoidal category, free on objects $\{\vdash p_1, \vdash p_2, \ldots\}$, where $\wedge$ is $\cap$ and $\mathbf{true}$ is the constant $\curlyvee$, whose objects are *intuitionistic assertive* formulas and whose morphisms $f\colon A\to B$ are equivalence classes of **AH-G1** sequent derivations (modulo permissible permutations of inferences). Similarly, we let $(\mathcal{B}, \vee, \mathbf{false})$ be the monoidal category, free on objects $\{\mathcal{H}\, p_1, \mathcal{H}\, p_2, \ldots\}$, where $\vee$ is $\curlyvee$ and **false** is the constant $\curlywedge$, whose objects are *co-intuitionistic hypothetical* formulas and whose morphisms $f\colon C\to D$ are equivalence classes of **AH-G1** sequent derivations (modulo permissible permutations of inferences). Consider Proposition 4.3 in Section 4: it gives the basic proof theoretic ingredients of the construction.

[7] In the context of 2-categories, the notation $\mathbf{B}^{op(0,1)}$ means that the *op* operation applies to 0-cells and 1-cells.

- The operations $\diamondminus\colon \mathcal{A} \to \mathcal{B}$ and $\boxdot\colon \mathcal{B} \to \mathcal{A}$ are adjoint functors between the cartesian category $(\mathcal{A}, \cap, \curlyvee)$ and the monoidal category $(\mathcal{B}, \curlyvee, \curlywedge)$.
- The proofs of Proposition 4.3 (ii) correspond to the construction of the unit and the co-unit of the adjunction.
- The contravariant monoidal functor $(\ \_\ )^*$ is simply the duality $(\ )^{a\perp_h}\colon \mathcal{A} \to \mathcal{B}^{op}$.
- We let $\langle A|C\rangle$ be the set of (equivalence classes of) derivations of $A \Rightarrow \boxdot C$.
- The cartesian category $(\mathcal{A}, \cap, \curlyvee)$ is in fact *cartesian closed*, i.e., exponents $A \supset B$ can be defined so that there is a natural bijection between $\mathcal{A}(M \wedge A, \boxdot B)$ and $\mathcal{A}(A, M \supset \boxdot B)$.
- The provable equivalences in Proposition 4.3 (iii) provide a "De Morgan" definition of intuitionistic implication in polarized bi-intuitionistic logic, i.e., a natural bijection between $\mathcal{A}(A, M \supset \boxdot B)$ and $\mathcal{A}(A, \boxdot((_j\frown \diamondminus M) \curlyvee B)$.
- By composing, we obtain the family of natural bijections

$$\chi_{M,A,B}\colon \langle M \wedge A|B\rangle \to \langle A|M^* \vee B\rangle.$$

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